Math 160: Leibniz and His Harmonic Triangle

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1 Biography:

Modern Calculus was fundamentally shaped by the self-taught mathematician, Gottfried Wilhelm Leibniz(1646-1716). He was born in Leipzig, Germany in 1646, and matured quickly under great influence from his father, who was a moral philosophy professor at the University of Leipzig. Unfortunately, at the age of 6, his father died and young Leibniz was left without direction except to follow in his father's footsteps and his mother's philosophy. It happened that his father had collected a large personal library covering numerous areas of interest and as a result of his death, these books were passed on to young Leibniz. He made use of the library from an early age and this nourished his already advanced capabilities.

Leibniz entered college early, at age fifteen, and received his bachelor's degree in philosophy, followed two years later by a law degree from the same university where his father taught. He continued this passion for higher education at the University of Leipzig, but his doctoral thesis on teaching law through a historical lens was not well received. It is possible that the academicians at the time were not enthusiastic about a student at his age promoting such a novel idea. Despite his perceived brilliance, he was nonetheless rejected from advancing in his academic career. However, Leibniz did not give up. He traveled to Nuremberg to attend the University of Altdorf instead, and finally obtained his doctorate in philosophy at the age 21.

After graduating, he got a job in a government-related position working with the Elector of Mainz (the representative of the region of power seated in Mainz during the period of the Holy

Roman Empire). He started focusing on mathematics in 1672 when he was sent to Paris on a diplomatic mission. During the following four years in Paris, he created his own version of calculus based on his own logical philosophy.

2 Harmonic Triangle:

Leibniz's interests in mathematics can be traced to a friend, Christiaan Huygens(1629-1695), one of the most famous Dutch scientists, whom he met during his four-year stay in Paris. Later, Huygens became his mentor, and heavily influenced Leibniz' direction in mathematics. It was Huygens who initially proposed one of the problems that focused Leibniz on triangular numbers, finding the sum of the infinite reciprocals of the triangular numbers. The infinite reciprocals of the triangular numbers are represented here:

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\cdots+=?$$

To tackle this problem, Leibniz soon recognized a trick; first factor each terms by 2, which had the effect of converting each denominator term into two consecutive numbers multiplied. From there he could use the telescoping sum technique to solve. Shown as the following:

$$\begin{aligned}
&\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\dots\\
=&2\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\dots\right)\\
=&2\left(\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\frac{1}{5\times 6}+\dots\right)\\
=&2\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}\dots\right)\\
=&2\times 1\\
=&2
\end{aligned}$$

Leibniz quickly solved Huygens' challenge. However, he was not satisfied, and a component of his philosophy about the world led him to believe that the properties of the series could be generalized. He believed that logical thinking could convert things into a universal form. He started to ponder such questions as,

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\frac{1}{56}+\cdots=?$$
$$\frac{1}{1}+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}+\frac{1}{126}+\cdots=?$$

The answer to the above questions led to the birth of the Harmonic Triangle, also honored with his name, the Leibniz Triangle. He constructed this particular triangle using recursion, where he denotes $L(r,1)=\frac{1}{r}$ as the first entry of each row, where r represents the index of the row, starting from 1. If he let c be the column number, which is also indexed from 1; he could find the value of each entry by such a relationship: $L(r,c)=L(r-1,c-1)-L(r,c-1)$. For example, by looking at the 4th row, second column, the calculation yields $\frac{1}{12}$. This can be calculated by taking the difference of the row above and one position left, relative to the current position, which yields $\frac{1}{3}$, and the entry one column to the left of the current position in the same row, that yields $\frac{1}{4}$.

							1							
						$\boxed{\frac{1}{2}}$		$\frac{1}{2}$						
					$\frac{1}{3}$		$\frac{1}{6}$		$\frac{1}{3}$					
				$\frac{1}{4}$		$\frac{1}{12}$		$\frac{1}{12}$		$\frac{1}{4}$				
			$\frac{1}{5}$		$\frac{1}{20}$		$\frac{1}{30}$		$\frac{1}{20}$		$\frac{1}{5}$			
		$\frac{1}{6}$		$\frac{1}{30}$		$\frac{1}{60}$		$\frac{1}{60}$		$\frac{1}{30}$		$\frac{1}{6}$		
	$\frac{1}{7}$		$\frac{1}{42}$		$\frac{1}{105}$		$\frac{1}{140}$		$\frac{1}{105}$		$\frac{1}{42}$		$\frac{1}{7}$	
$\frac{1}{8}$		$\frac{1}{56}$		$\frac{1}{168}$		$\frac{1}{280}$		$\frac{1}{280}$		$\frac{1}{168}$		$\frac{1}{56}$		$\frac{1}{8}$
			$\vdots$				$\vdots$				$\vdots$			

Tab. 1: Leibniz Triangle showing rows 1 through 8. The yellow highlight indicates characteristic 2 below, where the infinite sum of the elements in column 2 of every row add to the element above the beginning of the sum.

We can summarize several properties using this convenient Harmonic Triangle when compared to Pascal's Triangle:

Property 1: In Pascal Triangle, each entry is the sum of the two entries directly above it. In Leibniz's Triangle, each entry also can be reached by taking the sum of the two terms directly below it $L(r, c) = L(r + 1, c) + L(r + 1, c + 1)$.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & \vdots & & & & \vdots & & & & \vdots & & &
\end{array}
$$

Tab. 2: Pascal's Triangle, showing rows 0 through 7

Property 2: Both Triangle can be obtained using the Binomial coefficient. In Pascal Triangle we can express each entry as $\binom{r}{c}$, where r is the row number, and c is the column number, counting from 0 in both rows and columns. In the Harmonic Triangle, the entry $L(r, c)$ can be computed directly instead of doing the arithmetic recursively with its "neighbors". The direct form is written as $L(r, c) = \frac{1}{r\binom{r-1}{c-1}}$, with $r \geq c \geq 1$, and includes the binomial coefficient in the denominator.

Property 3: Both Triangles have symmetric entries. Their relation to the binomial coefficient is preserved through their symmetric structures.

Besides sharing some similar properties with Pascal's Triangle, the Harmonic Triangle also has its own unique characteristics such as the following:

Characteristic 1: Summing all the denominators of the nth row, produces the result: $n \cdot 2^{n-1}$.

Characteristic 2: Summing diagonally (or one can read as summing all the entries in the same column down the triangle), yields the number above the first leading one as $S_c = \sum_{r=c}^{\infty} L(r, c) = \frac{1}{c-1}$.

For c=2,3,4... For instance: looking at the infinite series along the second diagonal, starting from the third row (see Table 1):

$$\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\cdots=\frac{1}{2}$$

Note: here $\frac{1}{2}$ is the number directly above the leading term,$\frac{1}{6}$. This discover, indeed, solved and summarized the original challenge from Huygens that Leibniz had been working on. Beyond that, now he could solved all the sum of infinite reciprocals of Pascal number for each row as he aimed to. A quick check of the original challenge as shown below:

$$\begin{aligned}
&\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\frac{1}{56}+\dots\\
&=1+(\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\frac{1}{56}+\dots)\\
&=1+3(\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\frac{1}{105}+\dots)\\
&=1+3\cdot\frac{1}{6}\\
&=1+\frac{1}{2}\\
&=\frac{3}{2}
\end{aligned}$$

Characteristic 3: The 2 edges of the triangle from the top form the harmonic series: $1,\frac{1}{2},\frac{1}{3}\dots,\frac{1}{n},\dots$. As we all know that the sum of harmonic series diverges. An interesting fact is that that the alternative harmonic series converges, and $\sum_1^\infty(-1)^{n+1}\frac{1}{n}$ converges to 0.

Characteristic 4: The entry formula can be also written as:

$$L(r,c)=\int_0^1 x^{c-1}(1-x)^{r-c}dx$$

This is known as beta density function in Probability theory. Taking $r\geq c\geq 1$, we can utilize $\beta(r-c+1,c+1)$ by shifting $r\geq c\geq 0$. A well known fact is that the beta function is strongly connected with gamma function, which is connected with the factorial form of the Binomial formula. We recognize the general beta function as $\beta(u,v)=\int_0^1 x^{u-1}(1-x)^{v-1}dx$ where $u\geq 1, v\geq 1$.

Now, if we take the entry formula beginning at $\frac{1}{(r+1)\binom{r}{c}}$, we can produce the following:

$$\begin{aligned}\frac{1}{(r+1)\binom{r}{c}} &= \frac{1}{(r+1)\cdot\frac{r!}{(r-c)!c!}} \\ &= \frac{1}{\frac{(r+1)!}{(r-c)!c!}} \\ &= \frac{(r-c)!c!}{(r+1)!} \\ &= \frac{\Gamma(r-c+1)\cdot\Gamma(c+1)}{\Gamma(r+2)} \\ &= \int_0^1 x^{r-c}(1-x)^c dx \\ &= \beta(r-c+1, c+1)\end{aligned}$$

3 Calculus Development:

Creating this harmonic triangle to generalize a pattern made Leibniz more deeply interested in the subject of mathematics. He continued investigating taking the infinite sums and differences of sequences. His other well known work also consists of another triangle, the Characteristic Triangle, which he developed and which produces $\frac{\pi}{4}$ using geometric expansion.

$$\begin{aligned}\frac{\pi}{4} &= \int_0^1 \frac{1}{1+y^2}dy \\ &= \int_0^1 \frac{1}{1-(-y^2)}dy \\ &= \int_0^1 \left[1+(-y^2)+(-y^2)^2+(-y^2)^3\right]dy \\ &= 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...\end{aligned}$$

From here, we can see that his ideas shifted from discrete arithmetic calculations to the continuity of a circle (or a curvature figure). Devoting himself to his math work, he continually attempted to formulate general methods or algorithms that can apply to a much wider breadth of problems. He used the words, "...the world is governed by a 'non-causal relationship of harmony, parallelism, or correspondence.'" Thus, he rigorously pursued his lifelong project of inventing such a universal language-symbolic logic, which is a system of notation and that aims to simplify the math behind it, and provides a system for logical reasoning. In 1684-1686, he published a periodical article, *on*

Acta Eruditorum, in which he introduced the integration symbol,$\int$. In this series of papers, he also presented his idea for differential calculus. These works helped in develop the foundation for higher order differentials in his later work. A quick example is the use of chain rule, for example, if $z = f(y), y = g(x)$.

In Leibniz notation, this can be done by inserting the infinitesimal value dy as a term which can eventually be removed by cancellation: $\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$, where the infinitesimal values dy,dx are nonzero. Using his symbolic logic(notation) to understand would be much simpler than the method that Language used: $z'(x) = f'(g) \times g'(x)$

4 Conclusion:

In 1714, two years before he died, Leibniz wrote a paper *Historia et origo calculi differentialis* in which he substantially contributed to the creation of analytic calculus. From his early efforts in mathematics up to this point, we can say that it is his ability to form the analogy between the sum and difference of infinite series which lead to the creation of modern calculus. These generalizations have become a fundamental tool to understand many forms of knowledge from areas such as probability, physics, and philosophy, to name a few.

Both Leibniz and Newton made enormous contributions to the development of modern calculus and at roughly the same time. However, they took different approaches. For Newton, calculus was a more fundamental description of changing rates, the fluxion, which is an idea that directly relates to continuity. However, in Leibniz's approach, he emphasized the symbolic notation, which gives more logical structure and reasoning to the relationship between discrete infinitesimal incrementation. Thus, his integration has naturally become the infinite sum of differentials. Their different methods to approach this same universal idea became the step-stone to push many fields of science forward in the 17th and 18th centuries.

5 References

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